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On invariant linear functionals

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ABSTRACT

An equivalent condition is given for the existence of a strictly positive order continuous linear functional invariant under the adjoint of a positive order continuous operator on a Riesz space. A dual version of this condition gives an equivalent condition for the existence of a weak unit invariant under a positive operator on a perfect Banach lattice.

INTRODUCTION

In the ergodic theory of Markov operators important results, such as the pointwise ergodic theorem, depend upon the existence of a subinvariant equivalent measure. The purpose of this note is to generalize to the context of Riesz spaces these notions and an equivalent condition, proved in [6] by Neveu. Further references to the history of the subject (for Markov operators) appear in [3]. We will also apply these results by duality to find a criterion for the existence of (sub-) invariant weak units for a positive operator on a perfect Banach lattice, which was proved earlier differently.

A Markov operator is a positive contraction on a L_1 space of some σ-finite measure. It can also be defined as an order continuous positive contraction P on the dual space L_∞; this is equivalent to it being the adjoint of a positive contraction on L_1 (see [4], § 1). The reason for this is that by the Radon-Nikodym theorem L_1, apart from being the predual of L_∞, is also the space of

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all order continuous linear functionals on L_∞. This implies that the adjoint of P, which acts on the dual of L_∞, leaves L_1 invariant. A finite invariant measure is an element of L_1 that is a fixed point of the adjoint of P. When considered as a positive linear functional, this measure is equivalent to the given measure iff it is strictly positive (its value for any nonzero characteristic function is nonzero).

Let L be a Riesz space. We wish to generalize results from L_∞ to L. We will use the terminology of [7], [8], and [2]. Disjointness in L will be denoted by $\perp$, and for a subset A of L, A^d is the band $\{y \in L : y \perp x \ (x \in A)\}$.

The order dual $L^\sim$ of L is the space of all order bounded linear functionals on L. $L_n^\sim$ is the band of L consisting of the order continuous functionals. The null ideal of $0 \leq \nu \in L^\sim$ is the ideal $N_\nu := \{x \in L : \nu(|x|) = 0\}$. If $\varphi \in L_n^\sim$, then N_φ is a band in L. A functional $0 \leq \nu \in L^\sim$ is strictly positive if $N_\nu = \{0\}$. This is denoted $\nu \gg 0$.

P will always be an order continuous positive linear operator on L. Its order adjoint $P^\sim$ is defined on $L^\sim$ by $\langle x, P^\sim \nu\rangle = \langle Px, \nu\rangle$ for all $x \in L$, $\nu \in L^\sim$. Clearly $P^\sim(L_n^\sim) \subset L_n^\sim$. $0 \leq \nu \in L^\sim$ is called *invariant* under P if $P^\sim \nu = \nu$.

EXISTENCE OF INVARIANT FUNCTIONALS

Our main result is a generalization of a classical theorem that characterizes the existence of a finite invariant equivalent measure for a Markov operator. The proof uses Banach limits. First we need a lemma. It was suggested by the referee, and simplifies the original.

LEMMA 1. *Let L be an Archimedean Riesz space possessing a strictly positive order continuous linear functional λ. Let ν be a positive linear functional on L, and let φ be its projection on the band $L_n^\sim$. Then N_ν is an order dense ideal in N_φ. In particular $N_\varphi = \{0\}$ if and only if $N_\nu = \{0\}$.*

PROOF. Put $\psi := \nu - \varphi$. Then $0 \leq \psi \in L^\sim$, $\nu = \varphi + \psi$ and $\psi \perp \lambda$. By [8], Th. 90.5, $C_\psi \subset N_\lambda$. Since λ is strictly positive, we have $N_\lambda = \{0\}$, so $C_\psi = \{0\}$ and consequently N_ψ is an order dense ideal in L. This together with $N_\nu = N_\varphi \cap N_\psi$ implies that N_ν is order dense in N_φ.

A Banach limit is a positive linear functional LIM on l_∞, such that for all $(x_n) \in l_\infty$ we have $\mathrm{LIM}(x_n) = \mathrm{LIM}(x_{n+1})$, and $\mathrm{LIM}(x_n) \geq \liminf_n x_n$ (see for example [3] Ch. IV).

THEOREM 2. *Let L be an Archimedean Riesz space with a weak unit u and a strictly positive order continuous linear functional λ. Let P be an order continuous positive operator on L, with $Pu = u$. A necessary condition for the existence of a strictly positive order continuous linear functional φ invariant under P is*

(1) $\quad \liminf_n \langle P^n h, \lambda\rangle > 0$ *for every* $0 < h \in L$.

This condition is sufficient if also

(2) $\sup_n \langle P^n h, \lambda \rangle < \infty$ *for every* $h \in L$.

If L has a locally solid topology τ *such that* λ *is continuous and the set of operators* $\{P^n\}_{n=1}^{\infty}$ *is equicontinuous in* τ, *then condition* (2) *is satisfied and* φ *is continuous in* τ.

PROOF OF NECESSITY. Let $0 \ll \varphi \in L_n^{\sim}$ be P-invariant, and assume that for $h \in L^+$ one has

(*) $\liminf_n \langle P^n h, \lambda \rangle = 0.$

Since u is a weak unit, by taking $h \wedge u$ we can assume without loss of generality that $h \leq u$, and so for all n $P^n h \leq P^n u = u$. Now it is well known that λ, being strictly positive, is a weak unit in $L_n^{\sim}$. Thus $\varphi - (\varphi \wedge k\lambda) \downarrow_k 0$. For every $k = 1, 2, \ldots$ and $n = 1, 2, \ldots$

$$\langle P^n h, \varphi \rangle = \langle P^n h, \varphi \wedge k\lambda \rangle + \langle P^n h, \varphi - \varphi \wedge k\lambda \rangle \leq$$

$$\leq k \langle P^n h, \lambda \rangle + \langle u, \varphi - (\varphi \wedge k\lambda) \rangle.$$

We can fix k so that the right summand is arbitrarily small, and together with (*) we get $\liminf_n \langle P^n h, \varphi \rangle = 0$. But $P^{\sim} \varphi = \varphi$, so $\langle h, \varphi \rangle = 0$, and from $\varphi \gg 0$ follows $h = 0$.

PROOF OF SUFFICIENCY ASSUMING (2). (2) implies that the operator

$$T(h) := (\langle Ph, \lambda \rangle, \langle P^2 h, \lambda \rangle, \ldots)$$

maps L into l_∞. Take a Banach limit LIM on l_∞. Then $\nu := \mathrm{LIM} \circ T$ defines a positive linear functional on L. From the shift property of LIM we get

$$\langle h, P^{\sim} \nu \rangle = \langle Ph, \nu \rangle = \mathrm{LIM}(\langle P^{n+1} h, \lambda \rangle) = \mathrm{LIM}(\langle P^n h, \lambda \rangle) = \langle h, \nu \rangle$$

so $P^{\sim} \nu = \nu$. Let φ be the projection of ν on $L_n^{\sim}$. $P^{\sim} \varphi \leq P^{\sim} \nu = \nu$, and since $P^{\sim} \varphi \in L_n^{\sim}$, the maximality of the projection gives $P^{\sim} \varphi \leq \varphi$. Also $\langle u, P^{\sim} \varphi \rangle = \langle Pu, \varphi \rangle = \langle u, \varphi \rangle$, so from the order continuity of $\varphi - P^{\sim} \varphi$ which is ≥ 0 and the fact that u is a weak unit in L we get $P^{\sim} \varphi = \varphi$. We show now that $N_\varphi = \{0\}$. Otherwise there exists by Lemma 1 some $0 < h \in N_\nu$. But (1) gives

$$\langle h, \nu \rangle = \mathrm{LIM}(\langle P^n h, \lambda \rangle) \geq \liminf \langle P^n h, \lambda \rangle > 0$$

contradicting $h \in N_\nu$.

Let now τ be a locally solid topology in L such that λ is continuous and $\{P^n\}$ equicontinuous in τ. Then (2) is satisfied since λ is continuous and the set $\{P^n h\}$ is τ-bounded (For let V be any 0-neighborhood in τ. By equicontinuity there exists a 0-neighborhood V_1 with $P^n(V_1) \subset V$ for all n. h is absorbed by V_1, and thus $\{P^n h\}$ is absorbed by V). To show that T is τ-continuous, let $\varepsilon > 0$ be given. From the continuity of λ there exists a 0-neighborhood V in τ with $|\lambda(x)|<\varepsilon$ for all $x\in V$. By equicontinuity of $\{P^n\}$ there exists V_1 with $P^n(V_1)\subset V$ for all n, so for all $h\in V$ we get $\|Th\|_\infty<\varepsilon$. Now LIM is continuous on l_∞ (it is positive), so ν is continuous in τ. Finally φ is continuous in τ because $\varphi\leq\nu$ and L' is an ideal in L.

REMARKS. 1) The condition on the topology is easy to formulate when L is a normed lattice. It says then that $\sup_n \|P^n\|<\infty$. If L is also Banach, then λ is automatically continuous as a positive linear functional.

2) In proving the sufficiency we used $Pu=u$ merely to obtain from $P^{\sim}\varphi\leq\varphi$ equality. Hence if no such u exits (even if L has no weak unit at all) we still get a *sub*invariant $0\ll\varphi\in L_n^{\sim}$. The assumption $Pu=u$ is also not needed if L is a Banach lattice with an order continuous norm, since then ν itself, which is positive, is order continuous, and we do not have to use φ at all.

Suppose the assumptions of Theorem 2 hold. Then $Pu=u$ implies that the order dense ideal I_u generated by u is invariant under P. In Ch. IV of [3], culminating in Th. E and its corollaries, it is proved that for a Markov operator on L_∞ our condition (1) is equivalent to conditions (c)–(f) in the following Theorem 3. The method there is based on a lemma which in our notations states the following:

Let $0<f\leq u$ not satisfy condition (1). Then there exists $0<g\leq f$ and an infinite sequence $\{n_i\}$ such that $\sum_{i=1}^{\infty} P^{n_i}g\leq 2u$.

Using the lemma one readily obtains the existence of nonzero components υ of u for which $1/N\sum_{n=1}^{N} P^n\upsilon$ converges to 0 u-uniformly.

This method carries over with almost no change for the operator P on I_u, and Theorem 2 thus yields

THEOREM 3. *Under the assumptions of Theorem* 2 *including condition* (2) *the following are equivalent:*

(a) *There exists some* $0\ll\varphi\in L_n^{\sim}$ *invariant under* P.
(b) *For every* $0<f\in I_u$, $\liminf\langle P^nf,\lambda\rangle>0$.
(c) *For every* $0<f\in I_u$, $\liminf_N 1/N\sum_{n=1}^{N}\langle P^nf,\lambda\rangle>0$.
(d) *For every* $0<f\in I_u$, $\limsup_N 1/N\sum_{n=1}^{N}\langle P^nf,\lambda\rangle>0$.
(e) *For no* $0<f\in I_u$ *does* $1/N\sum_{n=1}^{N}P^nf$ *order-converge to zero as* $N\to\infty$.
(f) *For no* $0<f\in I_u$ *does* $1/N\sum_{n=1}^{N}P^nf$ *converge to zero u-uniformly as* $N\to\infty$.

One can also obtain a converse result:

THEOREM 4. *Under the assumptions of Theorem* 2 *including condition* (2) *the following are equivalent:*

(a) *There does not exist any nonzero invariant* $\varphi\in L_n^{\sim}$.

(b) *There exists a sequence of positive elements* $\{f_m\}$ *increasing to* u *such that, for all* m, $1/N \sum_{n=1}^{N} P^n f_m$ *converges to zero* u*-uniformly as* $N \to \infty$.

Application to invariant weak units.

For perfect Riesz spaces we can deduce the existence of subinvariant weak units from Theorem 2. Recall that a Riesz space L is called perfect if the embedding of L in $(L_n^\sim)_n^\sim$ is surjective, and thus L and $(L_n^\sim)_n^\sim$ are isomorphic.

THEOREM 5. *Let* L *be perfect Riesz space with a weak unit* υ. *Let* P *be positive (necessarily order continuous) on* L. *If the conditions*

(1) $\liminf_n \langle P^n \upsilon, \varphi \rangle > 0$ *for every* $0 < \varphi \in L_n^\sim$

(2) $\sup_n \langle P^n \upsilon, \varphi \rangle < \infty$ *for every* $\varphi \in L_n^\sim$

are satisfied, then there exists a weak unit u *in* L *with* $Pu \leq u$.

PROOF. P is a positive order continuous operator on $L_n^\sim$. The element υ, identified with its image in $(L_n^\sim)_n^\sim$, is strictly positive. So if we put in the sufficiency part of Theorem 2 (taking into account Remark 2) $L_n^\sim$ instead of L, $P^\sim$ instead of P and υ instead of λ, we obtain the existence of a $0 \ll u \in (L_n^\sim)_n^\sim = L$ with $Pu \leq u$. The strict positivity of u as a linear functional ensures now that it is a weak unit in L.

For a Riesz space endowed with a complete metrizable locally solid topology any positive linear functional is continuous. The topology is called Lebesgue if order convergence implies topological convergence (for Banach lattices this coincides with the definition of order continuous norm).

A Banach lattice with an order continuous norm is perfect if and only if every norm bounded increasing sequence has a supremum (cf. [2], Th. 22.2). Thus the next theorem constitutes for Banach lattices a proof different from [5], Cor. 1, and [1], Th. 2.3.

THEOREM 6. *Let* L *be a perfect Riesz space with a Lebesgue metrizable complete topology* τ. *Let* υ *be a weak unit in* L. *Let* P *be a positive operator on* L *such that* $\{P^n\}$ *is equicontinuous in* τ. *Then*

(1) $\liminf_n \langle P^n \upsilon, \varphi \rangle > 0$ *for every* $0 < \varphi \in L_n^\sim$

is a necessary and sufficient condition for the existence of a weak unit u *with* $Pu = u$.

PROOF OF SUFFICIENCY. Denote the dual space of L by L'. Since τ is metrizable complete and Lebesgue, $L' = L_n^\sim$ (the inclusion of L' in $L_n^\sim$ follows immediately from Lebesgueness, while the other inclusion is a consequence of the properties of τ by our previous remark).

As in the proof of Theorem 2 condition (2) of Theorem 5 is implied by the equicontinuity, so by Theorem 5 there exists a weak unit u_1 in L with $Pu_1 \leq u_1$. Let u be the infimum of the decreasing sequence $\{P^n u_1\}$, which exists since a perfect Riesz space is Dedekind complete. Since τ is Lebesgue we have $u = \lim P^n u_1$, and $Pu = u$. It remains to prove that u is a weak unit in L. Suppose, on the contrary that it is not. Then, as an element of $(L_n^{\sim})_n^{\sim}$, it is not strictly positive, so there exists some $0 < \varphi \in L_n^{\sim} = L'$ such that $\langle u, \varphi \rangle = 0$. For $k = 1, 2, \ldots$ and $n = 1, 2, \ldots$ we have

$$(*) \qquad \langle P^n v, \varphi \rangle = \langle P^n(v \wedge ku_1), \varphi \rangle + \langle P^n(v - v \wedge ku_1), \varphi \rangle \leq$$
$$\leq k \langle P^n u_1, \varphi \rangle + \langle P^n(v - v \wedge ku_1), \varphi \rangle.$$

Since u_1 is a weak unit, $v \wedge ku_1 \uparrow v$, which implies that the right term can be made arbitrarily small uniformly in n, by taking k large enough, as the topology is Lebesgue and $\{P^n\}$ is equicontinuous. Furthermore, $\langle P^n u_1, \varphi \rangle$ can be made small by taking n large, since $\lim P^n u_1 = u$ and $\langle u, \varphi \rangle = 0$. Thus we get from (*) $\lim \langle P^n v, \varphi \rangle = 0$ contradicting condition (1).

PROOF OF NECESSITY. Suppose that for some $0 \leq \varphi \in L'$ we have

$$(**) \qquad \liminf_n \langle P^n v, \varphi \rangle = 0.$$

Since v is a weak unit $u - (u \wedge kv) \downarrow_k 0$, and by Lebesgueness $u - (u \wedge kv) \xrightarrow{\tau} 0$. For $k = 1, 2, \ldots$ and $n = 1, 2, \ldots$ we have similarly to (*)

$$(***) \qquad \langle P^n u, \varphi \rangle \leq k \langle P^n v, \varphi \rangle + \langle P^n(u - u \wedge kv), \varphi \rangle.$$

Let $\varepsilon > 0$ be given. By continuity of φ there exists a 0-neighborhood V in τ with $|\langle y, \varphi \rangle| < \varepsilon$ whenever $y \in V$. By equicontinuity of $\{P^n\}$ there exists a 0-neighborhood V_1 in τ with $P^n(V_1) \subset V$ for all n. Choose k large enough so that $u - u \wedge kv \in V_1$. Then $\langle P^n(u - u \wedge kv), \varphi \rangle < \varepsilon$ for all n so (**) and (***) give $\liminf_n \langle P^n u, \varphi \rangle = 0$. But $Pu = u$ so $\langle u, \varphi \rangle = 0$, and since u is a weak unit in L and φ is order continuous we get $\varphi = 0$.

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